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On Spin(7) holonomy metric based on $\mathrm{SU}(3)/U(1)$: II

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Abstract

We continue the investigation of Spin(7) holonomy metric of cohomogeneity one with the principal orbit $\mathrm{SU}(3)/U(1)$. A special choice of $U(1)$ embedding in SU(3) allows more general metric ansatz with five metric functions. There are two possible singular orbits in the first-order system of Spin(7) instanton equation. One is the flag manifold $\mathrm{SU}(3)/T^2$ also known as the twistor space of $\mathbf{CP}(2)$ and the other is $\mathbf{CP}(2)$ itself. Imposing a set of algebraic constraints, we find a two-parameter family of exact solutions which have SU(4) holonomy and are asymptotically conical. There are two types of asymptotically locally conical (ALC) metrics in our model, which are distinguished by the choice of S^1 circle whose radius stabilizes at infinity. We show that this choice of M theory circle selects one of the possible singular orbits mentioned above. Numerical analyses of solutions near the singular orbit and in the asymptotic region support the existence of two families of ALC Spin(7) metrics: one family consists of deformations of the Calabi hyper-Kähler metric, the other is a new family of metrics on a line bundle over the twistor space of $\mathbf{CP}(2)$.

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1. Introduction

By string dualities intriguing dynamics in supersymmetric compactification of superstrings and M theory are often associated with singularities in manifolds of special holonomy which appear at finite distance in the moduli space. If the singularity is isolated and

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conical, we may expect that the details of the metric far from the singularity are irrelevant and as an approximation of the singular geometry take a simple Ricci-flat cone metric over an n-dimensional Einstein manifold M:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2\,\mathrm{d}\Omega^2, \quad 0 \le r < +\infty, \tag{1.1}$$

where the Einstein metric $\mathrm{d}\Omega^2$ satisfies $R_{ab} = (n-1)g_{ab}$. For supersymmetry a parallel spinor should exist on the cone and it comes from a Killing spinor on M [1–3]. Unless, the Einstein manifold is the n-dimensional sphere $S^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$, there is a conical singularity at $r = 0$. When the manifold M is a homogeneous space G/K, a resolution of the singularity may be provided by the following metric of cohomogeneity one [4–6]:

$$\mathrm{d}\tilde{s}^2 = \mathrm{d}t^2 + g_{G/K}(t), \quad t_0 \le t < +\infty, \tag{1.2}$$

where $g_{G/K}(t)$ is a t-dependent homogeneous metric on the principal orbit G/K. This sort of resolution of an isolated conical singularity has been employed recently in the discussion of IR strong coupling dynamics of supersymmetric gauge theories based on gauge theory/gravity correspondence in large N limit [7–14].

The requirement of special holonomy, which can be expressed as linear constraints on the spin connection ω_{ab}, gives a first-order system of flow equations for one-parameter family of homogeneous metrics $g_{G/K}(t)$. The boundary condition should be specified in solving the flow equation. At the boundary $t = t_0$ there appears a singular orbit G/H with $K \subset H \subset G$. This singular orbit has a finite volume and the original conical singularity is developed when the volume of G/H tends to vanishing. The coset H/K has to be a sphere S^k for the principal orbit G/K to degenerate smoothly to the singular orbit [15]. When there are several choices of the subgroup H such that $H/K \simeq S^k$, there may be more than one way of resolving the conical singularity. A famous example is given by the conifold that is a cone over the five-dimensional coset space $T^{1,1} = \mathrm{SU}(2) \times \mathrm{SU}(2)/U(1)$. There are three possible singular orbits [16]:

1. $H = U(2)$, $G/H \simeq S^2$, $H/K \simeq S^3$,
2. $H = \mathrm{SU}(2)$, $G/H \simeq S^3$, $H/K \simeq S^2$,
3. $H = U(1) \times U(1)$, $G/H \simeq S^2 \times S^2$, $H/K \simeq S^1$.

In this paper we will see a similar example of this kind, when the principal orbit is the seven-dimensional coset space $N(1,1) = \mathrm{SU}(3)/U(1)$.

The other side of the boundary is specified by the asymptotic behavior of the solution. A standard behavior is that the homogeneous metric $g_{G/K}(t)$ asymptotically approaches to the original Einstein metric $\mathrm{d}\Omega^2$. Such metrics are called asymptotically conical (AC). From the viewpoint of compactifications of M theory we are also interested in the asymptotic behavior called asymptotically locally conical (ALC) [17], where there is a circle whose radius remains finite at infinity. In [18], by considering the geometry of ALE fibration over a supersymmetric cycle, it has been argued that an M theoretic lift of a type IIA geometry with $D6$ branes wrapping on the SUSY cycle is given by purely gravitational configuration. (See also [19] for a relation of the M theoretic lift to Spin$_c$ structure.) Such $D6$ brane configurations have been discussed from the dual picture of eight-dimensional supergravity in [20,21]. Since the M theory circle which is related to the string coupling of IIA theory should remain finite asymptotically, the corresponding metric is expected to be ALC. In fact when the SUSY cycle is S^4 in Spin(7) manifold and S^3 in G_2 manifold, such ALC metrics were constructed in [17,22], respectively. More recently a similar ALC metric has been found for a SUSY cycle $\mathbf{CP}(2)$ in [19,23,24]. Even if we assume that the metric is ALC, the choice of M theory circle in the principal orbit may not be unique, when there are more than one irreducible modules of dimension one in the isotropy representation on the tangent space of the principal orbit. Due to the special choice of $U(1)$ subgroup to be introduced shortly, the isotropy representation of the coset space $N(1,1) = \mathrm{SU}(3)/U(1)$ has three one-dimensional irreducible components. Recently, M theory on ALC Spin(7) manifolds has been discussed in [19,25].

In this paper taking the homogeneous space $\mathrm{SU}(3)/U(1)$ (also known as the Aloff–Wallach space), we investigate aspects of the special holonomy metrics of cohomogeneity one. In our previous work [23] we left a choice of $U(1)$ subgroup in SU(3) free so that the triality $W(\mathrm{SU}(3))$ (=the Weyl group) symmetry was manifest. In the following we will fix the embedding so that the $U(1)$ subgroup is $\mathrm{diag}(\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{-2\mathrm{i}\theta})$. In this case we can make a more general metric ansatz with five functions, while in general the number is 4. In Section 2 we derive a first-order system for Spin(7) holonomy and classify a possible singular orbit appearing at the boundary. In our metric ansatz there is a natural candidate for a Kähler two form. The closedness (or the integrability) of the candidate two form gives a set of algebraic constraints that allows us to solve the flow equation exactly. In Section 3 we present a two-parameter family of exact solutions which is asymptotically conical. They are SU(4) holonomy metrics on the line bundle over the flag manifold $\mathrm{SU}(3)/T^2$, which is a two-sphere bundle over $\mathbf{CP}(2)$. When one of the parameters vanishes, then the S^2 fiber collapses and the metric reduces to the Calabi hyper-Kähler metric on $T^*\mathbf{CP}(2)$. An analysis of ALC solutions is carried out in Section 4. Due to the generalized metric ansatz with five metric functions there are two choices of a circle whose radius remains finite asymptotically. We find that if we assume the metric is non-singular, the ALC asymptotic behavior requires a reduction of the number of metric functions from 5 to 4, but the way of reduction depends on the choice of S^1 factor in the principal orbit. From the perturbative analysis around the singular orbit we see one of the possible singular orbits is selected by each reduction and thus there are two types of ALC Spin(7) metrics. The topology of the singular orbit and the choice of M theory circle cannot be independent and the singular orbit is either $\mathbf{CP}(2)$ or $\mathrm{Flag}_6 = \mathrm{SU}(3)/T^2$ depending on the choice of asymptotic M theory circle.

Since we could not find explicit solutions in general, we have numerically worked out perturbative series expansions both around the singular orbit and in the asymptotic region. In Section 5, based on this numerical analysis we propose the “moduli” space of Spin(7) metrics of cohomogeneity one with the principal orbit $\mathrm{SU}(3)/U(1)$ for the special choice of $U(1)$ subgroup. Especially, we observe that two types of ALC metrics in Section 4 are in fact interpolated by the exactly known Ricci-flat Kähler metrics obtained in Section 3. The existence of ALC metrics whose singular orbit is Flag_6 is shown only numerically. But their qualitative behavior is much like the Atiyah–Hitchin metric in four dimensions. Hence we believe that this is an analogue of Spin(7) metric called $\mathbf{C}_8$ in [24,26], whose singular orbit is $\mathbf{CP}(3)$, the twistor space of S^4.

2. Instanton equation with five metric functions

The maximal torus T^2 of the Lie group SU(3) is two-dimensional and its $U(1)$ subgroup is specified by integers $\vec{n} = (n_1, n_2, n_3)$ with $n_1 + n_2 + n_3 = 0$. Without loss of generality we can assume that there is no common divisor of n_i. Explicitly, the $U(1)$ subgroup is given by $\mathrm{diag}(\mathrm{e}^{\mathrm{i}n_1\theta}, \mathrm{e}^{\mathrm{i}n_2\theta}, \mathrm{e}^{\mathrm{i}n_3\theta})$. In previous paper on Spin(7) metric of cohomogeneity one with the principal orbit $\mathrm{SU}(3)/U(1)$, we took the following metric ansatz [23]:

$$g = \mathrm{d}t^2 + a(t)^2(\sigma_1^2 + \sigma_2^2) + b(t)^2(\Sigma_1^2 + \Sigma_2^2) + c(t)^2(\tau_1^2 + \tau_2^2) + f(t)^2 T_A^2, \tag{2.1}$$

which is consistent with any choice of the embedding parameters $\vec{n}$ and consequently gives a formulation which has manifest symmetry under $\Sigma_3 = W(\mathrm{SU}(3))$; the Weyl group of SU(3). Our convention of SU(3) left invariant one forms is summarized in Appendix A. The components of invariant one form for the maximal torus T^2 are denoted as T_A and T_B. The corresponding generators are given by

$$E_A = -\frac{1}{\Delta}\begin{pmatrix} \alpha_B & 0 & 0 \\ 0 & \beta_B & 0 \\ 0 & 0 & \gamma_B \end{pmatrix}, \qquad E_B = \frac{1}{\Delta}\begin{pmatrix} \alpha_A & 0 & 0 \\ 0 & \beta_A & 0 \\ 0 & 0 & \gamma_A \end{pmatrix} \tag{2.2}$$

with $\alpha_A + \beta_A + \gamma_A = \alpha_B + \beta_B + \gamma_B = 0$ and $\Delta = \beta_A\alpha_B - \alpha_A\beta_B$. The generator of the $U(1)$ subgroup is E_B. Note that the bases of the Lie algebra su(3) and the components of the left invariant one form are in dual relation and hence the role of parameters $\alpha_A, \beta_A, \gamma_A$ and $\alpha_B, \beta_B, \gamma_B$ is exchanged.

When the $U(1)$ subgroup generated by E_B decouples from one of σ, Σ and τ, more general metric ansatz is allowed since in this case the isotropy representation of $\mathrm{SU}(3)/U(1)$ becomes

$$\frac{\mathrm{su}(3)}{u(1)} = \mathbf{p}_1 \oplus \mathbf{p}_2 \oplus \mathbf{p}_3 \oplus \widetilde{\mathbf{p}_3} \oplus \mathbf{p}_4, \tag{2.3}$$

where $\dim \mathbf{p}_1 = \dim \mathbf{p}_2 = 2$ and $\dim \mathbf{p}_3 = \dim \widetilde{\mathbf{p}_3} = \dim \mathbf{p}_4 = 1$. Then the metric ansatz becomes

$$g = \mathrm{d}t^2 + a(t)^2(\sigma_1^2 + \sigma_2^2) + b(t)^2(\Sigma_1^2 + \Sigma_2^2) + c_1(t)^2\tau_1^2 + c_2(t)^2\tau_2^2 + f(t)^2 T_A^2, \tag{2.4}$$

where we assume that τ_1 and τ_2 are singlets. The reduction of the holonomy group from SO(8) to Spin(7) is represented by the octonionic instanton equation [27–29] (see also Appendix B). We can see the octonionic instanton equation derived from the above ansatz does not have T_B component, if and only if $\mathrm{d}\tau_i$ has no T_B component. We have $\nu_B = 0$ and hence $\alpha_A = \beta_A$ (see Appendix A). Then the generator of the $U(1)$ subgroup is $E_B = \mathrm{diag}(1, 1, -2)$ and the charge vector $\vec{n}$ in the Maurer–Cartan equation of $\mathrm{d}T_A$ is fixed to be $\vec{n} = (1, 1, -2)$. Note that this is the case where the action of the Weyl group degenerates. We obtain the following system of first-order differential equations as the octonionic instanton equation on the spin connection ω_{ab} derived from the metric ansatz (2.4):

$$\frac{\dot a}{a}=\frac{b^2+c_1^2-a^2}{2abc_1}+\frac{b^2+c_2^2-a^2}{2abc_2}-\frac{f}{a^2},$$
$$\frac{\dot b}{b}=\frac{c_1^2+a^2-b^2}{2abc_1}+\frac{c_2^2+a^2-b^2}{2abc_2}-\frac{f}{b^2},$$
$$\frac{\dot c_1}{c_1}=\frac{a^2+b^2-c_1^2}{abc_1}+\frac{2f}{c_1c_2}+\frac{c_2^2-c_1^2}{2c_1c_2f},$$
$$\frac{\dot c_2}{c_2}=\frac{a^2+b^2-c_2^2}{abc_2}+\frac{2f}{c_1c_2}+\frac{c_1^2-c_2^2}{2c_1c_2f},$$
$$\frac{\dot f}{f}=\frac{f}{a^2}+\frac{f}{b^2}-\frac{2f}{c_1c_2}+\frac{(c_1-c_2)^2}{2c_1c_2f}. \tag{2.5}$$

This first-order system has a discrete $\mathbf{Z}_2\times\mathbf{Z}_2$ symmetry generated by $(a,b,c_1,c_2)\to(\pm a,\mp b,-c_1,-c_2)$ and two exchange symmetries $a\leftrightarrow b$ and $c_1\leftrightarrow c_2$. We note that though any independent sign flip of metric functions that is not necessarily included above has no effect on the metric itself or at the level of Ricci-flatness, but do *not* keep the instanton equation invariant. The first-order system (2.5) is an integral of the second-order Einstein equation and the change in the instanton equation means the different ways of integration.

Let us classify possibilities of the singular orbit compatible with the evolution equation (2.5). Group theoretically the singular orbit is in one to one correspondence to a subgroup H which satisfy $U(1)\subset H\subset \mathrm{SU}(3)$ and $H/U(1)$ should be a sphere which is collapsing at the singular orbit. Thus we find three possibilities:

1. $H=U(1)\times U(1)$. In this case $H/U(1)\simeq S^1$ is collapsing and the singular orbit is the twistor space of $\mathbf{CP}(2)$; $\mathrm{SU}(3)/H\simeq \mathrm{Flag}_6$, which is topologically a two sphere bundle over $\mathbf{CP}(2)$.
2. $H=\mathrm{SU}(2)$. In this case $H/U(1)\simeq S^2$ is collapsing and the singular orbit is $\mathrm{SU}(3)/H\simeq S^5$, which is the Hopf bundle over $\mathbf{CP}(2)$.
3. $H=S(U(2)\times U(1))$. In this case $H/U(1)\simeq S^3$ is collapsing and the singular orbit is $\mathrm{SU}(3)/H\simeq\mathbf{CP}(2)$ itself.

We assume that the singular orbit is at $t=0$ and make the following series expansion for small t:

$$a(t)=p+\sum_{k\ge1}a_kt^k,\qquad b(t)=q+\sum_{k\ge1}b_kt^k,\qquad c_1(t)=m+\sum_{k\ge1}c_{1k}t^k,$$
$$c_2(t)=n+\sum_{k\ge1}c_{2k}t^k,\qquad f(t)=r+\sum_{k\ge1}f_kt^k. \tag{2.6}$$

The parameters p,q,m,n,r are regarded as the “initial conditions” at the singular orbit.

Substituting the series expansion to (2.5) and looking at the leading order, we obtain

$$a_1 = \frac{1}{2}\left(\frac{q}{m} + \frac{m}{q} - \frac{p^2}{mq} + \frac{q}{n} + \frac{n}{q} - \frac{p^2}{nq} - \frac{2r}{p}\right),$$
$$b_1 = \frac{1}{2}\left(\frac{p}{m} + \frac{m}{p} - \frac{q^2}{mp} + \frac{p}{n} + \frac{n}{p} - \frac{q^2}{np} - \frac{2r}{q}\right),$$
$$c_{11} = \frac{p}{q} + \frac{q}{p} - \frac{m^2}{pq} + \frac{2r}{n} + \frac{n}{2r} - \frac{m^2}{2nr},$$
$$c_{21} = \frac{p}{q} + \frac{q}{p} - \frac{n^2}{pq} + \frac{2r}{m} + \frac{m}{2r} - \frac{n^2}{2mr},$$
$$f_1 = \frac{r^2}{p^2} + \frac{r^2}{q^2} - \frac{2r^2}{mn} + \frac{n}{2m} + \frac{m}{2n} - 1. \tag{2.7}$$

Now the above three possibilities of the singular orbit correspond, respectively, to the following initial conditions:

1. S^1 is collapsing; $r = 0$,
2. S^2 is collapsing; $p = 0$, or $q = 0$,
3. S^3 is collapsing; $p = r = 0$, or $q = r = 0$.

Firstly, in case 2 there is no regular solution at the singular orbit, since there is no way to make f_1 regular.[1] On the other hand, in case 1 we see that the regularity of c_{11} and c_{21} requires $m^2 = n^2$. But $m = n$ implies $f_1 = 0$, which means that the S^1 is collapsing "too" fast near $t = 0$. Thus only the case $m = -n$ can give non-singular solutions. This also means that this type of singular orbit is not allowed in generic cases where we have $c_1 = c_2$. In this case p and q are free parameters and $f_1 = -2$. Finally, in case 3 there are non-singular solutions if $m^2 = n^2 = q^2$ or $m^2 = n^2 = p^2$. Thus, near $t = 0$ we have two types of boundary conditions which correspond to cases 1 and 3, respectively,

$$g \to \mathrm{d}t^2 + 4t^2 T_A^2 + p^2(\sigma_1^2 + \sigma_2^2) + q^2(\Sigma_1^2 + \Sigma_2^2) + m^2(\tau_1^2 + \tau_2^2), \tag{2.8}$$

$$g \to \mathrm{d}t^2 + t^2(T_A^2 + \sigma_1^2 + \sigma_2^2) + m^2(\Sigma_1^2 + \Sigma_2^2 + \tau_1^2 + \tau_2^2). \tag{2.9}$$

Note that the terms with σ_i^2, Σ_i^2 and τ_i^2 in (2.8) describe the singular orbit Flag_6 squashed by the parameters p, q and m, while the term with $\Sigma_i^2 + \tau_i^2$ in (2.9) represents the singular orbit $\mathbf{CP}(2)$ with the Fubini–Study metric. In the following we call the first case A-type boundary and the second case B-type boundary. If we regard the homogeneous space Flag_6 as an S^2-bundle over $\mathbf{CP}(2)$, then B-type boundary may be reduced from A-type one by making the fiber S^2 collapse. The higher order terms of the series expansion are summarized in Appendix C.

[1] However, another special choice of $U(1)$ embedding seems to allow S^5 as a singular orbit [24]. It might be very interesting to see why it is the case, since there is no odd-dimensional SUSY cycle in eight dimensions.

3. Explicit AC solutions of SU(4) holonomy

In terms of the vielbeins (the orthonormal frames) of our metric ansatz, we can write down the following non-degenerate two form

$$\omega = f\,\mathrm{d}t \wedge T_A - c_1 c_2 \tau_1 \wedge \tau_2 - a^2 \sigma_1 \wedge \sigma_2 - b^2 \Sigma_1 \wedge \Sigma_2, \tag{3.1}$$

which is a natural candidate for a Kähler form. Using the first-order equation for functions a, b and c_i, we see that the condition $\mathrm{d}\omega = 0$ is equivalent to the constraints

$$a^2 + b^2 + c_1 c_2 = 0, \qquad c_1 + c_2 = 0. \tag{3.2}$$

They are compatible with the first-order system (2.5) and we obtain the following reduction with $c := c_1 = -c_2$:

$$\dot{a} = -\frac{f}{a}, \qquad \dot{b} = -\frac{f}{b}, \qquad \dot{c} = -\frac{2f}{c}, \tag{3.3}$$

$$\dot{f} = \frac{f^2}{a^2} + \frac{f^2}{b^2} + \frac{2f^2}{c^2} - 2. \tag{3.4}$$

We can solve this reduced first-order system exactly. The first two equations of (3.3) implies

$$b^2 - a^2 = \ell^2, \tag{3.5}$$

where ℓ^2 is an integration constant. Due to the exchange symmetry of a and b, we may assume $b^2 - a^2 \geq 0$. Furthermore by a change of variables $\mathrm{d}t = c/(2f)\,\mathrm{d}r$ we can integrate a, b, c to obtain

$$a^2 = \tfrac{1}{2}(r^2 - \ell^2), \qquad b^2 = \tfrac{1}{2}(r^2 + \ell^2), \qquad c^2 = r^2, \tag{3.6}$$

where we have fixed an integration constant by requiring $c^2 = r^2$. Substituting the above solution into Eq. (3.4) we have

$$\frac{\mathrm{d}}{\mathrm{d}r} f^2 = 2r - 2f^2 \left(\frac{r}{r^2 - \ell^2} + \frac{r}{r^2 + \ell^2} + \frac{1}{r} \right). \tag{3.7}$$

It is possible to integrate this differential equation:

$$f^2 = \frac{r^2}{4} \left(1 - \frac{\ell^4}{r^4} \right) U(r), \qquad U(r) = 1 - \frac{k^8}{(r^4 - \ell^4)^2}. \tag{3.8}$$

We thus find the following metric of SU(4) holonomy:

$$\begin{aligned} \mathrm{d}s^2 = {} & \left(1 - \frac{\ell^4}{r^4} \right)^{-1} U(r)^{-1}\,\mathrm{d}r^2 + \frac{1}{2}(r^2 - \ell^2)(\sigma_1^2 + \sigma_2^2) + \frac{1}{2}(r^2 + \ell^2)(\Sigma_1^2 + \Sigma_2^2) \\ & + r^2(\tau_1^2 + \tau_2^2) + \frac{1}{4} r^2 \left(1 - \frac{\ell^4}{r^4} \right) U(r) T_A^2, \quad (k^4 + \ell^4)^{1/4} \leq r, \end{aligned} \tag{3.9}$$

which is asymptotically conical. The singular orbit at $r = (k^4 + \ell^4)^{1/4}$ ($k \neq 0$) is the flag manifold $\mathrm{SU}(3)/T^2$, or the twistor space of $\mathbf{CP}(2)$. Hence this metric is a Ricci-flat Kähler metric on the canonical line bundle over the flag manifold and it is in the class discussed in [30,31]. (See also [32] on the construction of Ricci-flat metrics on the canonical line bundle over Hermitian symmetric spaces based on the Kähler potential of supersymmetric gauge theory.) We also note that when $\ell = 0, k \neq 0$, the metric constructed in [16] is reproduced. The first-order system in [16] corresponds to the case $a = b, c_1 = -c_2$ in this paper and cannot cover the most general case. This is the reason why we can obtain more general solutions with two parameters. On the other hand, when $\ell \neq 0, k = 0$ then $U(r) \equiv 1$ and the solution reduces to the Calabi hyper-Kähler metric over $T^*\mathbf{CP}(2)$ of Sp(2) holonomy [16,33]:

$$\begin{aligned} \mathrm{d}s^2 &= \left(1 - \frac{\ell^4}{r^4}\right)^{-1} \mathrm{d}r^2 + \frac{1}{2}(r^2 - \ell^2)(\sigma_1^2 + \sigma_2^2) + \frac{1}{2}(r^2 + \ell^2)(\Sigma_1^2 + \Sigma_2^2) \\ &\quad + r^2(\tau_1^2 + \tau_2^2) + \frac{1}{4}r^2\left(1 - \frac{\ell^4}{r^4}\right)T_A^2, \quad \ell \leq r \end{aligned} \tag{3.10}$$

with three Kähler forms,

$$\begin{aligned} &\omega^1 = \omega, \qquad \omega^2 = c\,\mathrm{d}t \wedge \tau_1 - cfT_A \wedge \tau_2 - ab(\sigma_1 \wedge \Sigma_1 - \sigma_2 \wedge \Sigma_2), \\ &\omega^3 = c\,\mathrm{d}t \wedge \tau_2 + cfT_A \wedge \tau_1 + ab(\sigma_1 \wedge \Sigma_2 + \sigma_2 \wedge \Sigma_1). \end{aligned} \tag{3.11}$$

The flag manifold $\mathrm{SU}(3)/T^2$ is a two-sphere bundle over $\mathbf{CP}(2)$ and in the limit $k \to 0$ the S^2-fiber collapses to develop the singular orbit $\mathbf{CP}(2)$.

4. Two types of ALC Spin(7) metric

From the view point of compactification of M theory it is of great interest to classify possible ALC metric. When $c_1 = c_2$, an example of such ALC Spin(7) metric is given by [23,24]

$$\begin{aligned} \mathrm{d}s^2 &= \frac{(r-\ell)^2}{(r+\ell)(r-3\ell)}\mathrm{d}r^2 + (r-\ell)(r+\ell)(\sigma_1^2 + \sigma_2^2 + \Sigma_1^2 + \Sigma_2^2) \\ &\quad + (r-3\ell)(r+\ell)(\tau_1^2 + \tau_2^2) + \ell^2\frac{(r+\ell)(r-3\ell)}{(r-\ell)^2}T_A^2, \quad 3\ell \leq r, \end{aligned} \tag{4.1}$$

where the fiber over the base $\mathbf{CP}(2)$ is not $\mathbf{R}^4$ but $\mathbf{R}^4/\mathbf{Z}_2$. Note that this is different from the case of the Calabi metric. This is due to the fact that for the Calabi metric it is $\sigma_1^2 + \sigma_2^2$ part that is collapsing at the singular orbit, but it is $\tau_1^2 + \tau_2^2$ in the above ALC metric.

Let us consider the ALC Spin(7) solutions that interpolate between the short distance geometry of the form $\mathbf{R}^2 \times \mathrm{Flag}_6$ (A-type boundary) or $\mathbf{R}^4 \times \mathbf{CP}(2)$ (B-type boundary) and the large distance geometry of the form $S^1 \times C(\mathrm{Flag}_6)$, where $C(\mathrm{Flag}_6)$ is the seven-dimensional cone over Flag_6. We then assume that the metric functions take the following form for the large distance t:

1. $a(t) = ta_0 + \alpha(t)$, $b(t) = tb_0 + \beta(t)$, $c_1(t) = \gamma_1(t)$, $c_2(t) = tc_{20} + \gamma_2(t)$, $f(t) = tf_0 + \zeta(t)$,

2. $a(t) = ta_0 + \alpha(t)$, $b(t) = tb_0 + \beta(t)$, $c_1(t) = tc_{10} + \gamma_1(t)$, $c_2(t) = tc_{20} + \gamma_2(t)$, $f(t) = \zeta(t)$,

where a_0, b_0, c_{i0}, f_0 are constants and $\alpha, \beta, \gamma_i, \zeta$ are smooth functions tending to finite value for $t \to \infty$. In case 1, the S^1 direction is τ_1 and at large t the function c_1 approaches a constant $M_1 = \gamma_1(\infty)$, while in case 2 the function f of the S^1 direction T_A approaches a constant $M_2 = \zeta(\infty)$. The octonionic instanton equation (2.5) requires the following conditions on the leading coefficients:

$$a_0^2 = b_0^2 = 1, \qquad a_0 b_0 c_{20} = 1, \qquad f_0 = -\frac{1}{2} \quad \text{for case 1}, \tag{4.2}$$

$$a_0^2 = b_0^2 = 1, \qquad a_0 b_0 c_{20} = 1, \qquad c_{10} = c_{20} \quad \text{for case 2}. \tag{4.3}$$

Note that they are different only in the last condition, but this difference produces significant change as we will see shortly. Thus the possible cone metrics on $C(\mathrm{Flag}_6)$ consistent with the instanton equation are given by

$$g_c = \mathrm{d}t^2 + t^2\left(\sigma_1^2 + \sigma_2^2 + \Sigma_1^2 + \Sigma_2^2 + \tau_2^2 + \frac{T_A^2}{4}\right), \tag{4.4}$$

$$g_c' = \mathrm{d}t^2 + t^2(\sigma_1^2 + \sigma_2^2 + \Sigma_1^2 + \Sigma_2^2 + \tau_1^2 + \tau_2^2), \tag{4.5}$$

corresponding to (4.2) and (4.3), respectively. It follows that the boundary condition for the ALC metric is

$$g \to M_1^2\tau_1^2 + g_c \quad \text{or} \quad M_2^2 T_A^2 + g_c' \quad \text{for } t \to \infty. \tag{4.6}$$

We call the first case $A_\infty^\pm$ and the second case B_∞, and prove the following proposition. The sign $\pm$ corresponds to the choice $a_0 = \pm b_0$ in (4.2);[2] this difference does not appear at the level of cone metrics, but it must be distinguished at the level of instanton solutions.

Proposition. *If there exists a regular ALC solution interpolating between* $S^1 \times C(\mathrm{Flag}_6)$ *and* $\mathbf{R}^2 \times \mathrm{Flag}_6$ *or* $\mathbf{R}^4 \times \mathbf{CP}(2)$, *then the following holds*:

1. *For the boundary* $A_\infty^\pm$, $a(t) = \pm b(t)$ *in the whole region* $0 \le t \le \infty$ *and the solution approaches* $\mathbf{R}^2 \times \mathrm{Flag}_6$ *for* $t \to 0$.
2. *For the boundary* B_∞, $c_1(t) = c_2(t)$ *in the whole region* $0 \le t \le \infty$ *and the solution approaches* $\mathbf{R}^4 \times \mathbf{CP}(2)$ *for* $t \to 0$.

Remark. The part $\mathrm{d}t^2 + 4t^2 T_A^2$ in the metric (2.8) looks like

$$\mathrm{d}t^2 + t^2\,\mathrm{d}\psi^2, \quad 0 \le \psi < 4\pi, \tag{4.7}$$

when we fix the coordinates on Flag_6 in A-type boundary [23]. Therefore, the range of ψ must be adjusted to be that of usual polar coordinates on $\mathbf{R}^2$, $0 \le \psi < 2\pi$. This

[2] In (4.3), the choice of the sign does not make difference due to the $\mathbf{Z}_2$ symmetry of the instanton equation.

means the manifold in the boundary $A_\infty^{\pm}$ is $\mathrm{Flag}_6/\mathbf{Z}_2$ rather than Flag_6, which would have $0 \leq \psi < 4\pi$. While in the case of B_∞ it is not necessary to do such a modification since

$$\mathrm{d}t^2 + t^2(T_A^2 + \sigma_1^2 + \sigma_2^2) \tag{4.8}$$

in (2.9) is the standard metric on $\mathbf{R}^4$ written by the polar coordinates when we fix the coordinates on $\mathbf{CP}(2)$.

Proof. We first consider the case A_∞^+. From the instanton equation we have

$$a(t) - b(t) = N \exp\left(\int^t u(t')\,\mathrm{d}t'\right), \tag{4.9}$$

where N is an integration constant and

$$u(t) = \frac{1}{2ab}\left(\frac{c_1^2 - (a+b)^2}{c_1} + \frac{c_2^2 - (a+b)^2}{c_2} + 2f\right). \tag{4.10}$$

Suppose that a regular solution exists in the form (4.9) with $N \neq 0$. By using (4.2) with $a_0 = b_0$, it is easy to see the asymptotic behavior $a - b \simeq \mathrm{e}^{-2t/M_1}$ for $t \to \infty$. Note that the constant M_1 is required to be positive for the exponentially small suppression. If the solution approaches the singular orbit Flag_6, then the product c_1c_2 must be negative by the result of Section 2 (see also (C.1)). On the other hand, c_1c_2 is positive in the asymptotic region since $c_1c_2 \to M_1$ for $t \to \infty$ and hence c_1 or c_2 becomes zero at a certain time t_0, which contradicts the regularity condition. If the solution approaches the singular orbit $\mathbf{CP}(2)$, f is positive as seen in (C.2), so the negative f in the asymptotic region leads to a contradiction. In the case of (C.3), the product c_1c_2 is negative for $t \to 0$, which contradicts the sign in the asymptotic region. Thus $a = b$ in the whole region, and if there exists a regular solution with the boundary A_∞^+, then it approaches $\mathbf{R}^2 \times \mathrm{Flag}_6$ for $t \to 0$ since this boundary is only one consistent with $a = b$. Furthermore, by the discrete symmetry of the instanton equation there exists a regular solution with $a = -b$ for the boundary A_∞^-.

Next let us consider the case B_∞. By the instanton equation we have

$$c_1(t) - c_2(t) = N \exp\left(\int^t v(t')\,\mathrm{d}t'\right) \tag{4.11}$$

with

$$v(t) = \frac{4f^2 - (c_1+c_2)^2}{2c_1c_2f} - \frac{c_1+c_2}{ab}. \tag{4.12}$$

If we assume a regular solution (4.11) with $N \neq 0$, then similar arguments lead to a contradiction. Thus $c_1 = c_2$ in the whole region and the solution approaches $\mathbf{R}^4 \times \mathbf{CP}(2)$ given by (C.2) for $t \to 0$. □

5. Evidence for new global solutions

It is not easy to find exact solutions in general and we turn to numerical computations to examine the existence of global solutions to the octonionic instanton equation (2.5). The result of our analysis is summarized in Fig. 1, which shows possible lines for existence of global solutions in the two-dimensional parameter space of initial conditions at the singular orbit. The circle $p^2+q^2=m^2$ represents the Ricci-flat Kähler metrics of SU(4) holonomy obtained in Section 3. As one can see from (3.9), near the singular orbit at $r_0=(k^4+\ell^4)^{1/4}$ we have

$$g \to \mathrm{d}\rho^2 + 4\rho^2 T_A^2 + \tfrac{1}{2}(r_0^2+\ell^2)(\sigma_1^2+\sigma_2^2) + \tfrac{1}{2}(r_0^2-\ell^2)(\Sigma_1^2+\Sigma_2^2) + r_0^2(\tau_1^2+\tau_2^2), \tag{5.1}$$

where $\rho^2=r_0(r-r_0)/2$. By comparing with the expansion (C.1), we see that they are parametrized by the circle with radius $m=r_0$ in the (p,q)-space of A-type boundary. The four points on the circle, $(p,q)=(\pm m,0)$ and $(0,\pm m)$, correspond the Calabi hyper-Kähler metric given by (3.10), where the holonomy group is further reduced to Sp(2). Note that the Calabi metric satisfies B-type boundary and hence the singular orbit changes from Flag_6 to $\mathbf{CP}(2)$ at these points. The wavy lines attaching to the Calabi metric are the ALC metrics of Spin(7) holonomy whose existence was expected by the second statement of proposition (B_∞ boundary). Indeed, from the numerical analysis we can find non-singular solutions interpolating between $S^1\times C(\mathrm{Flag}_6)$ and $\mathbf{R}^4\times\mathbf{CP}(2)$ for the parameter region $q_1<-2/3$ of B-type boundary (C.2), with $q_1=-2/3$ giving the Calabi metric [23,24].

Finally, we discuss the new metrics of Spin(7) holonomy depicted by the lines $p=\pm q$, $p^2+q^2>m^2$ in Fig. 1, which we shall denote by $\mathbf{C}_8^*$. They are an analogue of Spin(7) metrics $\mathbf{C}_8$ on the line bundle over $\mathbf{CP}(3)$ discussed in [24,26]. Although we have not been able to find the solutions in closed form, the following arguments indicate they must

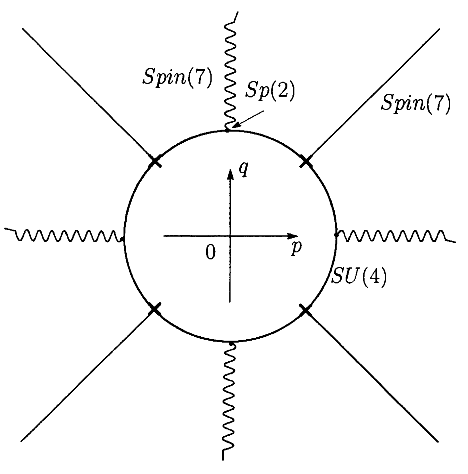

Fig. 1. Possible lines for the existence of global metric of special holonomy.

exist. The solutions on the two lines $p = q$ and $p = -q$ are related to each other by the action of the discrete symmetry of the instanton equation (2.5), and so we will consider the case $p = q$. By rescaling the parameter $p \to mp$, the perturbative expansion for A-type boundary becomes

$$\begin{aligned} a(t) &= b(t) = m\left(p + \frac{6p^2-1}{4p^3}\left(\frac{t}{m}\right)^2 + \cdots\right), \\ c_1(t) &= m\left(1 + \frac{2p^2-1}{2p^2}\left(\frac{t}{m}\right) + \frac{12p^4-4p^2+3}{8p^4}\left(\frac{t}{m}\right)^2 + \cdots\right), \\ c_2(t) &= -m\left(1 - \frac{2p^2-1}{2p^2}\left(\frac{t}{m}\right) + \frac{12p^4-4p^2+3}{8p^4}\left(\frac{t}{m}\right)^2 + \cdots\right), \\ f(t) &= -2t\left(1 - \frac{12p^4+20p^2-1}{12p^4}\left(\frac{t}{m}\right)^2 + \cdots\right), \end{aligned} \tag{5.2}$$

which shows the reduction $a = b$ of the instanton equation. If we put $c_3 \equiv -2f$, the first-order system with $a = b$ reduction is described by

$$\begin{aligned} \frac{\dot a}{a} &= \frac{c_1}{2a^2} + \frac{c_2}{2a^2} + \frac{c_3}{2a^2}, \qquad \frac{\dot c_1}{c_1} = -\frac{c_1}{a^2} + \frac{c_1^2-(c_2-c_3)^2}{c_1c_2c_3}, \\ \frac{\dot c_2}{c_2} &= -\frac{c_2}{a^2} + \frac{c_2^2-(c_3-c_1)^2}{c_1c_2c_3}, \qquad \frac{\dot c_3}{c_3} = -\frac{c_3}{a^2} + \frac{c_3^2-(c_1-c_2)^2}{c_1c_2c_3}. \end{aligned} \tag{5.3}$$

After the rescaling $a \to \sqrt{2}a$ we obtain exactly the same first-order system as in [24, Eq. (8)], where it was shown numerically that the solution with the boundary (5.2) is regular and ALC provided that the parameter p is chosen to satisfy $p^2 > 1/2$. It is easy to check the boundary $p^2 = 1/2$ corresponds to the AC solution (3.9) with the special value $\ell = 0$. Thus two-parameter family of ALC metrics $\mathbf{C}_8^*$ has the same topology as the canonical line bundle over Flag_6. The large distance geometry of $\mathbf{C}_8^*$ can be worked out as follows. By the proposition in Section 4, $\mathbf{C}_8^*$ approaches the boundary A_∞^+ for $t \to \infty$. After some calculation we find that the asymptotic expansion up to order t^{-3} is given by

$$\begin{aligned} a(t) &= t\left(1 + \frac{3}{8}\left(\frac{M}{t}\right)^2 + \frac{1}{4}\left(\frac{M}{t}\right)^3 + \frac{1}{2}\left(\frac{7}{64} - P\right)\left(\frac{M}{t}\right)^4 + \cdots\right), \\ c_1(t) &= M\left(1 - \frac{1}{2}\left(\frac{M}{t}\right)^2 - \frac{1}{2}\left(\frac{M}{t}\right)^3 + \cdots\right), \\ c_2(t) &= c_3(t) = t\left(1 - \frac{1}{2}\left(\frac{M}{t}\right) + P\left(\frac{M}{t}\right)^4 + \cdots\right). \end{aligned} \tag{5.4}$$

It should be noticed that the equality $c_2 = c_3$ is valid for all orders, if we assume that they can be expanded as power series in t^{-1}. Hence, the series coincides with the asymptotic form of the ALC solutions found in [17]. The parameters M, P correspond to m, p in the expansion around the singular orbit. Since the product $c_1 c_2 = -m^2$ for $t \to 0$, we must have $M < 0$. This sign is consistent with the exponentially small correction of the asymptotic expansion. Indeed, we have

$$\dot{c}_2 - \dot{c}_3 = (c_2 - c_3)\left(\frac{(c_2+c_3)^2 - c_1^2}{c_1 c_2 c_3} - \frac{c_2+c_3}{a^2}\right), \tag{5.5}$$

which leads to the asymptotic behavior $c_2 - c_3 \simeq \mathrm{e}^{4t/M}$ using the expansion (5.4), and the metric functions behave similarly to those in the Atiyah–Hitchin metric [26,34].

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Appendix A. Convention of SU(3) Maurer–Cartan forms

We use the following SU(3) Maurer–Cartan equation that is Σ_3 symmetric:

$$\begin{aligned}
\mathrm{d}\sigma_1 &= \Sigma_1 \wedge \tau_1 - \Sigma_2 \wedge \tau_2 + \kappa_A T_A \wedge \sigma_2 + \kappa_B T_B \wedge \sigma_2,\\
\mathrm{d}\sigma_2 &= -\Sigma_1 \wedge \tau_2 - \Sigma_2 \wedge \tau_1 - \kappa_A T_A \wedge \sigma_1 - \kappa_B T_B \wedge \sigma_1,\\
\mathrm{d}\Sigma_1 &= \tau_1 \wedge \sigma_1 - \tau_2 \wedge \sigma_2 + \mu_A T_A \wedge \Sigma_2 + \mu_B T_B \wedge \Sigma_2,\\
\mathrm{d}\Sigma_2 &= -\tau_1 \wedge \sigma_2 - \tau_2 \wedge \sigma_1 - \mu_A T_A \wedge \Sigma_1 - \mu_B T_B \wedge \Sigma_1,\\
\mathrm{d}\tau_1 &= \sigma_1 \wedge \Sigma_1 - \sigma_2 \wedge \Sigma_2 + \nu_A T_A \wedge \tau_2 + \nu_B T_B \wedge \tau_2,\\
\mathrm{d}\tau_2 &= -\sigma_1 \wedge \Sigma_2 - \sigma_2 \wedge \Sigma_1 - \nu_A T_A \wedge \tau_1 - \nu_B T_B \wedge \tau_1,\\
\mathrm{d}T_A &= 2\alpha_A \sigma_1 \wedge \sigma_2 + 2\beta_A \Sigma_1 \wedge \Sigma_2 + 2\gamma_A \tau_1 \wedge \tau_2,\\
\mathrm{d}T_B &= 2\alpha_B \sigma_1 \wedge \sigma_2 + 2\beta_B \Sigma_1 \wedge \Sigma_2 + 2\gamma_B \tau_1 \wedge \tau_2.
\end{aligned} \tag{A.1}$$

This form of the Maurer–Cartan equation is symmetric under the (cyclic) permutation of $(\sigma_i, \Sigma_i, \tau_i)$. From the Jacobi identity we see that the parameters $\alpha, \beta, \gamma, \kappa, \mu, \nu$, which describe the “coupling” of the Cartan generators $\{T_A, T_B\}$ satisfy

$$\begin{aligned}
&\alpha_A + \beta_A + \gamma_A = 0, \qquad \alpha_B + \beta_B + \gamma_B = 0, \qquad \kappa_A = \frac{1}{\Delta}(\beta_B - \gamma_B),\\
&\kappa_B = -\frac{1}{\Delta}(\beta_A - \gamma_A), \qquad \mu_A = -\frac{1}{\Delta}(\alpha_B - \gamma_B), \qquad \mu_B = \frac{1}{\Delta}(\alpha_A - \gamma_A),\\
&\nu_A = \frac{1}{\Delta}(\alpha_B - \beta_B), \qquad \nu_B = -\frac{1}{\Delta}(\alpha_A - \beta_A)
\end{aligned} \tag{A.2}$$

with $\Delta = \beta_A\alpha_B - \alpha_A\beta_B$ leaving four free parameters $(\alpha_{A,B}, \beta_{A,B})$. We may further put the "orthogonality" conditions:

$$\alpha_A\alpha_B + \beta_A\beta_B + \gamma_A\gamma_B = 0, \qquad \kappa_A\kappa_B + \mu_A\mu_B + \nu_A\nu_B = 0, \tag{A.3}$$

which reduces one parameter.

Appendix B. Reduction of holonomy group and the octonionic instanton equation

One of the ways to realize the reduction of holonomy group is to impose appropriate linear relations on the so(n) valued spin connection one form $\omega_{ab} = -\omega_{ba}$. It is rather amusing that in all the three cases which are most relevant from the viewpoint of M theory compactifications the expected number of linear relations is always 7, since dim SO(8) − dim $Spin(7)$ = dim SO(7) − dim G_2 = dim SO(6) − dim SU(3) = 7. There are topological relations behind this dimension counting; Spin(7)/$G_2 \simeq$ SO(8)/SO(7) $\simeq S^7$ and G_2/SU(3) $\simeq$ SO(7)/SO(6) $\simeq S^6$. In fact the following octonionic instanton equation gives a "master" equation for seven conditions required for the reduction [27,28].

$$\omega_{ab} = \tfrac{1}{2}\Psi_{abcd}\omega_{cd}, \tag{B.1}$$

where totally anti-symmetric tensor Ψ_{abcd} is defined by the structure constants of octonions ψ_{abc} as follows:

$$\Psi_{abc0} = \psi_{abc}, \quad 1 \leq a, b, c, \dots, \leq 7), \qquad \Psi_{abcd} = -\frac{1}{3!}\epsilon_{abcdefg}\psi_{efg}. \tag{B.2}$$

A conventional choice of the structure constants is

$$\psi_{abc} = +1 \quad \text{for} \quad (abc) = (123), (516), (624), (435), (471), (572), (673). \tag{B.3}$$

It can be shown that (B.1) implies the four form defined by

$$\Omega = \frac{1}{4!}\Psi_{abcd}\, \mathrm{e}^a \wedge \mathrm{e}^b \wedge \mathrm{e}^c \wedge \mathrm{e}^d \tag{B.4}$$

is closed and the metric has Spin(7) holonomy [29]. In the above convention of the structure constants of octonions the explicit form of the octonionic instanton equation is

$$\begin{aligned}
&\omega_{14} + \omega_{25} + \omega_{36} + \omega_{07} = 0, \qquad &&\omega_{71} + \omega_{62} + \omega_{35} + \omega_{04} = 0,\\
&\omega_{47} + \omega_{65} + \omega_{23} + \omega_{01} = 0, \qquad &&\omega_{67} + \omega_{12} + \omega_{54} + \omega_{03} = 0,\\
&\omega_{73} + \omega_{51} + \omega_{24} + \omega_{06} = 0, \qquad &&\omega_{57} + \omega_{46} + \omega_{31} + \omega_{02} = 0,\\
&\omega_{72} + \omega_{16} + \omega_{43} + \omega_{05} = 0.
\end{aligned} \tag{B.5}$$

If we simply substitute $\omega_{0k}(1 \leq k \leq 7)$, then (B.5) gives the seven conditions for G_2 holonomy. Further putting $\omega_{7j}(1 \leq j \leq 6)$ gives the seven conditions for SU(3) holonomy. We should emphasize that compared with the condition on the Riemann curvature, the condition on the spin connection depends on the gauge or the choice of coordinate system and therefore it is only a sufficient but not necessary condition for special holonomy.

Appendix C. Perturbative expansion around singular orbits

For A-type boundary the instanton equation is perturbatively solved in the form

$$\begin{aligned}
a(t) &= p + \left(\frac{1}{p} + \frac{(p^2+q^2-m^2)(p^2-q^2+m^2)}{4pq^2m^2}\right)t^2 + \cdots, \\
b(t) &= q + \left(\frac{1}{q} + \frac{(p^2+q^2-m^2)(-p^2+q^2+m^2)}{4p^2qm^2}\right)t^2 + \cdots, \\
c_1(t) &= m + \left(\frac{p^2+q^2-m^2}{2pq}\right)t \\
&\quad + \left(\frac{2}{m} - \frac{m(p^2+q^2-m^2)}{2p^2q^2} - \frac{(p^2+q^2-m^2)^2}{8p^2q^2m}\right)t^2 + \cdots, \\
c_2(t) &= -m + \left(\frac{p^2+q^2-m^2}{2pq}\right)t \\
&\quad - \left(\frac{2}{m} - \frac{m(p^2+q^2-m^2)}{2p^2q^2} - \frac{(p^2+q^2-m^2)^2}{8p^2q^2m}\right)t^2 + \cdots, \\
f(t) &= -2t\left(1 + \left(\frac{p^4+q^4+m^4-10p^2m^2-10q^2m^2-14p^2q^2}{12p^2q^2m^2}\right)t^2 + \cdots\right).
\end{aligned} \tag{C.1}$$

We note that the power series solution are completely fixed by the "initial conditions" p, q, m. (This should be compared with the case of B type boundary condition in the following.) The reduction $c_1 = -c_2$ is reproduced by imposing $p^2 + q^2 = m^2$ and the reduction $a = \pm b$ by $p = \pm q$.

There are two possible solutions for B-type boundary. One of these is given by

$$\begin{aligned}
a(t) &= t\left(1 - \frac{1}{2}(q_1+1)\left(\frac{t}{m}\right)^2 + \cdots\right), \qquad b(t) = m\left(1 + \frac{1}{2}\left(\frac{t}{m}\right)^2 + \cdots\right), \\
c_1(t) &= c_2(t) = m\left(1 + \left(\frac{t}{m}\right)^2 + \cdots\right), \qquad f(t) = t\left(1 + q_1\left(\frac{t}{m}\right)^2 + \cdots\right).
\end{aligned} \tag{C.2}$$

The other solution has the following expansion:

$$\begin{aligned}
a(t) &= t\left(1 - \frac{1}{6}\left(\frac{t}{m}\right)^2 + \cdots\right), \qquad b(t) = m\left(1 + q_2\left(\frac{t}{m}\right)^2 + \cdots\right), \\
c_1(t) &= m\left(1 + \left(\frac{t}{m}\right)^2 + \cdots\right), \qquad c_2(t) = -m\left(1 + 2(1-q_2)\left(\frac{t}{m}\right)^2 + \cdots\right), \\
f(t) &= t\left(-1 + \frac{2}{3}\left(\frac{t}{m}\right)^2 + \cdots\right).
\end{aligned} \tag{C.3}$$

These solutions include the free parameters q_1 and q_2 in addition to the scaling parameter m. In particular, both solutions with $q_1 = -2/3$ and $q_2 = 1/2$ lead to a same metric and this is in fact precisely the Calabi hyper-Kähler metric on $T^*\mathbf{CP}(2)$.

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